

# Infinite dimensional Riemannian symmetric spaces with fixed-sign curvature operator

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## Abstract

We associate to any Riemannian symmetric space (of finite or infinite dimension) a  $L^*$ -algebra, under the assumption that the curvature operator has a fixed sign.  $L^*$ -algebras are Lie algebras with a pleasant Hilbert space structure. The  $L^*$ -algebra that we construct, is a complete local isomorphism invariant and allows us to classify Riemannian symmetric spaces with fixed-sign curvature operator. The case of nonpositive curvature is emphasized.

## 1 Introduction

### 1.1 Riemannian symmetric spaces

At the very end of the nineteenth century and during the beginning of the twentieth century, E. Cartan did a famous work of classification. He began by completing the proof (by W. Killing) of the classification of complex semisimple Lie algebras during his Ph.D. thesis and he continued by classifying real semisimple Lie algebras. Some years later, he introduced the so-called Riemannian symmetric spaces (“*Une classe remarquable d’espaces de Riemann*”) and classified them. The classification of symmetric spaces was reminiscent of the classification of real forms of complex semisimple Lie algebras (see [Bor01]).

Infinite dimensional differential geometry grew up from the nineteen-twenties and it is not difficult to define when a Riemannian manifold, that is a manifold modeled on a separable Hilbert space with a Riemannian metric, is a symmetric space. Let  $(M, g)$  be a Riemannian manifold, a *symmetry* at a point  $p$  is an involutive isometry  $\sigma_p: M \rightarrow M$  such that  $\sigma_p(p) = p$  and the differential at  $p$  is  $-\text{Id}$ . If the exponential map at  $p$  is surjective then such a map is unique. A *Riemannian symmetric space* is a Riemannian manifold such that, at each point, there exists a symmetry and the exponential map is surjective.

An idea to classify these spaces could be to associate a “semisimple” Lie algebra to them, to classify infinite dimensional semisimple Lie algebras and then return to symmetric spaces. We do not know a general classification of infinite dimensional Lie algebras

nor a good notion of semisimple Lie algebras. Nonetheless, there is a remarkable exception to this lack of classification. R. Schue introduced complex  $L^*$ -algebras (Lie algebras with a compatible structure of Hilbert space, see Section 2) and classified the separable ones in [Sch60, Sch61]. Later, independently, V.K. Balachandran [Bal72], P. de la Harpe [dlH71] and I. Unsain [Uns71] classified separable real  $L^*$ -algebras.

Each  $L^*$ -algebra is an orthogonal sum of an abelian ideal and a semisimple ideal. Each separable semisimple  $L^*$ -algebra is a Hilbertian sum of simple ones. The simple  $L^*$ -algebras of infinite dimension belong to a finite list with three infinite families. They are closure of an increasing union of simple Lie algebras of finite dimension and classical type.

Unfortunately, the Lie algebra of the isometry group of a Riemannian symmetric space has no reason to be a  $L^*$ -algebra. For example, consider the Riemannian symmetric space  $P^2(\infty) \simeq GL_\infty^2(\mathbb{R})/O^2(\infty)$ , that is the space of positive invertible operators of some separable real Hilbert space, which are Hilbert-Schmidt perturbations of the identity. This space is an infinite dimensional generalization of the symmetric space  $SL_n(\mathbb{R})/SO_n(\mathbb{R})$  (See [dlH72, III.2] and [Lar07]). The full orthogonal group  $O(\infty)$  acts isometrically by conjugation on  $P^2(\infty)$ . In particular, the Lie algebra of all bounded skew-symmetric operators is a subalgebra of the Lie algebra of the isometry group. It is naturally a Banach Lie algebra but not a  $L^*$ -algebra.

*Remark 1.1.* It seems to be known that the isometry group of a Riemannian space is a Banach Lie group but we do not know any reference. In the sequel, we do not use this result and the Lie algebra of Killing fields will play the role of the Lie algebra of the isometry group. In finite dimension, the Lie algebra of the isometry group of a Riemannian symmetric space and the algebra of Killing fields are naturally isomorphic.

In the following theorem, we show that if one looks at a smaller (but large enough to encode the curvature tensor) Lie algebra, one can find a  $L^*$ -algebra. We refer to section 3.2 for the definition of the curvature operator.

**Theorem 1.2.** *Let  $(M, g)$  be a Riemannian symmetric space and let  $p$  be a point in  $M$ . If  $M$  has a fixed-sign curvature operator then there exists a real  $L^*$ -algebra  $L$  with an orthogonal decomposition*

$$L = \mathfrak{k} \oplus \mathfrak{p}$$

*which has the following properties :*

- (i) *the subspace  $\mathfrak{k}$  is a  $L^*$ -subalgebra of  $L$  and  $\mathfrak{p}$  is isometric to the tangent space  $T_p M$ ,*
- (ii) *the Lie algebra generated by  $\mathfrak{p}$  is dense in  $L$  and is isomorphic to a subalgebra of the Lie algebra of Killing fields on  $M$ .*

The  $L^*$ -algebra obtained in Theorem 1.2 is the only one which satisfies properties (i) and (ii) (see Lemma 3.2). We call it a  $L^*$ -algebra associated to  $(M, g)$ . Moreover, it allows us to give a complete description of Riemannian symmetric spaces up to local isomorphism.

**Theorem 1.3.** *Let  $(M, g)$  and  $(M', g')$  be Riemannian symmetric spaces with fixed-sign curvature operator. Let  $L, L'$  be  $L^*$ -algebras associated to  $M$  and  $M'$  as in Theorem 1.2. If there exists an isomorphism of  $L^*$ -algebras between  $L$  and  $L'$  which intertwines the orthogonal decompositions  $L = \mathfrak{k} \oplus \mathfrak{p}$  and  $L' = \mathfrak{k}' \oplus \mathfrak{p}'$  then  $M$  and  $M'$  are locally isomorphic.*

If the curvature operator of a Riemannian manifold is nonpositive (respectively nonnegative) then the sectional curvature is nonpositive (respectively nonnegative) but the converse is false in general (See, e.g., [GM75, §1.3]). In finite dimension, a Riemannian symmetric space has nonpositive (respectively nonnegative) curvature operator if and only if it has nonpositive (respectively nonnegative) sectional curvature. This fact holds because the curvature tensor is encoded in the Killing form of the Lie algebra of the isometry group (See [Sim62, Theorem 6], [GM75, Section 4] or Equation (3.2)). It is natural to ask whether the same is true in infinite dimension. More generally, we have the following question.

**Question 1.4.** Is it true that for any Riemannian symmetric space, there is an orthogonal decomposition of the tangent space  $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+$  such that  $\mathfrak{p}_-, \mathfrak{p}_0$  and  $\mathfrak{p}_+$  are commuting Lie triple systems and the restrictions of the curvature operator are nonnegative on  $\mathfrak{p}_-$ , vanishes on  $\mathfrak{p}_0$  and is nonpositive on  $\mathfrak{p}_+$ ?

A positive answer to this question would imply a complete classification of simply connected separable Riemannian symmetric spaces. Actually, if a symmetric space has a dense increasing sequence of totally geodesic subspaces of finite dimension then Proposition 3.4 shows that the answer to the above question is positive. Moreover, subsequent theorems will show that such a decomposition of the tangent space will imply the existence of a dense increasing sequence of totally geodesic subspaces of finite dimension.

Technics that we used in nonpositive curvature and nonnegative curvature are slightly different. In nonpositive curvature the Cartan-Hadamard theorem simplifies the classification and we give this simpler proof even if the technics used in nonnegative curvature are more general.

## 1.2 Nonpositive curvature

We now specialize to the case where the symmetric space has nonpositive curvature operator. Let  $M$  be a symmetric space with nonpositive curvature operator. We say that  $M$  has *noncompact type* if the  $L^*$ -algebra associated to  $M$  does not contain a nontrivial abelian ideal in  $\mathfrak{p}$ .

As in finite dimension, Proposition 4.1 shows that a Riemannian symmetric space of noncompact type is simply connected and a complete CAT(0) space.

**Definition 1.5.** Let  $(X_i, d_i)$  be a countable family of metric spaces with base points  $x_i \in X_i$ . The product  $\prod_i X_i$  is defined to be the set of elements  $y = (y_i)$  of the Cartesian

product of  $X_i$ 's such that  $\sum d(x_i, y_i)^2 < \infty$  and the distance between  $y = (y_i)$  and  $z = (z_i)$  is defined by  $d(y, z)^2 = \sum d(y_i, z_i)^2$ . This metric space is called the *Hilbertian product* of the spaces  $X_i$ .

This definition depends on the choice of base points but if each  $X_i$  has a transitive group of isometries then it does not depend on this choice (up to isometry). Moreover, it is complete if and only if each  $(X_i, d_i)$  is so.

*Remark 1.6.* In general, there is no notion (in the category of Riemannian manifolds) of Hilbertian product of Riemannian manifolds. The sectional curvature at each point has to be bounded (the Riemann 4-tensor at each point is continuous and thus the sectional curvature is bounded). For example, the Hilbertian products of spheres of radius  $1/n$  cannot be a Riemannian manifold such that each sphere embeds as a totally geodesic submanifold.

**Theorem 1.7.** *Let  $(M, g)$  be a separable Riemannian symmetric space of noncompact type then  $(M, g)$  is isometric to a Hilbertian product*

$$M \simeq \prod_i^2 M_i$$

where each  $M_i$  is an irreducible finite dimensional Riemannian symmetric space of noncompact type or is homothetic to an element of the following list :

$$\begin{aligned} GL_\infty^2(\mathbb{R})/O_\infty^2(\infty), \quad U^{*2}(\infty)/Sp^2(\infty), \quad U^2(p, \infty)/U^2(p) \times U^2(\infty), \quad O^2(p, \infty)/O^2(p) \times O^2(\infty) \\ O^{*2}(\infty)/U^2(\infty), \quad Sp_\infty^2(\mathbb{R})/U^2(\infty), \quad Sp^2(p, \infty)/Sp^2(p) \times Sp^2(\infty), \\ GL_\infty^2(\mathbb{C})/U^2(\infty), \quad O_\infty^2(\mathbb{C})/O^2(\infty), \quad Sp_\infty^2(\mathbb{C})/Sp^2(\infty) \end{aligned}$$

where  $p \in \mathbb{N} \cup \{\infty\}$ .

The elements of the previous list are hence the irreducible infinite dimensional Riemannian symmetric spaces of noncompact type. Their construction is described in Section 4.2.

The *rank* of a metric space is the supremum of dimensions of Euclidean spaces isometrically embedded. The paper [Duc12] was focused on some irreducible infinite dimensional Riemannian symmetric spaces of nonpositive sectional curvature with finite rank. For brevity, the following notation was used in [Duc12] :  $X_p(\mathbb{K})$  ( $p \in \mathbb{N}$ ) denotes the symmetric space  $O^2(p, \infty)/O^2(p) \times O^2(\infty)$ ,  $U^2(p, \infty)/U^2(p) \times U^2(\infty)$  or  $Sp^2(p, \infty)/Sp^2(p) \times Sp^2(\infty)$  depending on whether  $\mathbb{K}$  is the field of real, complex or quaternionic numbers. Actually, these spaces are the only ones to have infinite dimension and finite rank.

**Corollary 1.8.** *Let  $(M, g)$  be a separable Riemannian symmetric space of noncompact type. The rank of  $M$  is equal to its telescopic dimension. Moreover, if it is finite then*

$$M \simeq \prod_{i=1}^k M_i$$

where  $M_i$  is an irreducible finite dimensional Riemannian symmetric space of noncompact type or is homothetic to some  $X_p(\mathbb{K})$ .

The telescopic dimension of a CAT(0) space is a notion of dimension at large scale introduced in [CL10].

We conclude this section with an example of a space which is symmetric and has nonpositive curvature but which is not a Riemannian symmetric space. This is a purely infinite dimensional phenomenon. Let  $(X, d)$  be metric space. We say that  $X$  is a *CAT(0) symmetric space* if it is a complete CAT(0) space such that for any point  $x \in X$ , there exists an involutive isometry  $\sigma_x$  with unique fixed point  $x$ . Observe that this condition implies that  $x$  is the midpoint of  $y$  and  $\sigma_x(y)$  for any  $y \in X$ . In finite dimension, [CM09, Theorem 1.1] implies that any proper CAT(0) symmetric space is the product of a Euclidean space and a Riemannian symmetric space of noncompact type (and finite dimension). This theorem uses the solution to Hilbert's fifth problem and local compactness is crucial.

Let  $\mathbb{H}$  be the hyperbolic plane with sectional curvature -1 and let  $o$  be a point in  $\mathbb{H}$ . We set  $L^2([0, 1], \mathbb{H})$  to be the space of measurable maps  $f: [0, 1] \rightarrow \mathbb{H}$  such that  $\int d(f(t), o)^2 dt < \infty$ . This space is a CAT(0) symmetric space but not a Riemannian manifold, see Section 4.3.

### 1.3 Nonnegative curvature

In the case of nonnegative curvature, some more technicalities appear. The first one is the lack of automatic simply connectedness and the second one is the fact that the exponential map is not necessarily a diffeomorphism. Let  $M$  be a symmetric space with nonnegative curvature operator. We say that  $M$  has *compact type* if the  $L^*$ -algebra associated to  $M$  does not contain a nontrivial abelian ideal in  $\mathfrak{p}$ . Under the assumption of simply connectedness, we obtain the following theorem.

**Theorem 1.9.** *Let  $(M, g)$  be a simply-connected separable Riemannian symmetric space of compact type then  $(M, g)$  is isometric to a Hilbertian product*

$$M \simeq \prod_i^2 M_i$$

where each  $M_i$  is a simply-connected irreducible Riemannian symmetric space completely determined by the orthogonal symmetric  $L^*$ -algebra  $L_i$  associated to it. Each  $M_i$  can be finite dimensional or  $L_i$  is one of those orthogonal symmetric  $L^*$ -algebras described in Proposition 5.3.

### 1.4 Comments

W. Kaup obtained a classification of Hermitian symmetric spaces in [Kau81, Kau83]. His work uses the so-called Jordan-Hilbert algebras (Jordan algebras with a compatible

structure of Hilbert space and an adjoint map  $X \mapsto X^*$ ). His technics seem difficult to adapt to the real case. This approach of symmetric space of W. Kaup is closer to the one of O. Loos than the one of E. Cartan. The paper [Tum09] shows a description in terms of  $L^*$ -algebras of the irreducible Hermitian symmetric spaces.

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## 2 $L^*$ -algebras

**Definition 2.1.** A  $L^*$ -algebra is a Lie algebra with a structure of (complex or real) Hilbert space such that there is a  $\mathbb{R}$ -linear involution  $x \mapsto x^*$  satisfying, for all  $x, y, z$ , the equation

$$\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle. \quad (2.1)$$

A  $L^*$ -algebra is *semisimple* if  $\overline{[L, L]} = L$  and it is *simple* if it has no (closed and  $*$ -invariant) nontrivial ideal. A  $L^*$ -algebra is of *compact type* if it is semisimple and  $x^* = -x$  for all  $x$ . A semisimple  $L^*$ -algebra is of *noncompact type* if it has no ideal of compact type. An *isomorphism* between  $L^*$ -algebras is an isomorphism of Lie algebras that is also an isometry and intertwines the involutions.

**Example 2.2.** Let  $\mathcal{H}$  be a separable Hilbert space over  $\mathbb{K} = \mathbb{R}, \mathbb{C}$  or the field of quaternions. The Lie algebra endowed with the involution given by the adjoint and the Hilbert structure coming from the Hilbert-Schmidt scalar product is a  $L^*$ -algebra, which we denote by  $\mathfrak{gl}_\infty^2(\mathbb{K})$ . A choice of a Hilbert base for  $\mathcal{H}$  provides embeddings of the simple algebras of traceless operators  $\mathfrak{gl}_n(\mathbb{K})$  on  $\mathbb{K}^n$ , into  $\mathfrak{gl}_\infty^2(\mathbb{K})$  such that their increasing union is dense. The other examples of separable simple  $L^*$ -algebras are constructed in a similar way.

**Example 2.3.** Let  $\mathfrak{g}$  be a semisimple real Lie algebra of finite dimension. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition of  $\mathfrak{g}$ . The Killing form  $B$  of  $\mathfrak{g}$  is negative definite on  $\mathfrak{k}$  and positive definite on  $\mathfrak{p}$ . Moreover for any  $X, Y, Z$ , we have  $B([X, Y], Z) = -B(Y, [X, Z])$ . Hence, if we define  $(K + P)^* = -K + P$  (with  $K \in \mathfrak{k}$  and  $P \in \mathfrak{p}$ ) and  $\langle X, Y \rangle = B(X, Y^*)$  then  $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$  is a  $L^*$ -algebra. Actually, the map  $X \mapsto X^*$  is just the opposite of the Cartan involution.

Any separable  $L^*$ -algebra can be written as the direct sum of an Abelian ideal and a Hilbertian sum (as described below) of simple ideals. Moreover, simple  $L^*$ -algebras have been classified in the complex and real cases (see references in the introduction). The simple separable infinite dimensional real  $L^*$ -algebras of noncompact and compact types are recalled respectively in sections 4.2 and 5.

Let  $\{\mathcal{H}_i\}$  be a countable family of separable (real, complex or quaternionic) Hilbert spaces. The *Hilbertian sum* of this family, which we will denote by  $\oplus^2 \mathcal{H}_i$ , is the set of

sequences  $v = (v_i)$  such that  $\sum_i \|v_i\|^2$  is finite (see [Bou87, V.2.1]). Endowed with the inner product  $\langle u, v \rangle = \sum_i \langle u_i, v_i \rangle$ , the space  $\oplus^2 \mathcal{H}_i$  is also a separable Hilbert space.

**Proposition 2.4.** *Let  $(L_i)$  be a countable family of semisimple  $L^*$ -algebras such that there exists  $C \geq 0$  with  $\|\text{ad}(x)\| \leq C\|x\|$  for all  $i$  and all  $x \in L_i$ . For  $x = (x_i), y = (y_i) \in \oplus^2 L_i$ , set  $[x, y] = ([x_i, y_i])$  and  $x^* = (x_i^*)$ . Endowed with this structure, the Hilbertian sum  $\oplus^2 L_i$  is a  $L^*$ -algebra.*

*Proof.* Let  $(x_i) \in \oplus^2 L_i$  and  $y = (y_i) \in \oplus^2 L_i$  then  $[x, y] = \sum [x_i, y_i]$  is an element of  $\oplus^2 L_i$  since  $\|[x, y]\|^2 \leq \sum C^2 \|x_i\|^2 \|y_i\|^2 \leq C^2 \|x\|^2 \|y\|^2$ . This also shows that  $\text{ad}(x)$  is a linear bounded operator and the Lie bracket is also continuous. Continuity arguments show that  $\oplus^2 L_i$  is a Lie algebra and for all  $x \in \oplus^2 L_i$ ,  $\text{ad}(x)^* = \text{ad}(x^*)$ . Since  $L_i$  is semisimple, the equation (2.1) implies that  $\langle u_i, v_i^* \rangle = \langle v_i, u_i^* \rangle$  for all  $u_i \in L_i$  and  $v_i \in [L_i, L_i]$  (see [Sch60, Preliminaries]). Since  $\overline{[L_i, L_i]} = L_i$ , we have  $\|u_i^*\| = \|u_i\|$  for any  $u_i \in L_i$ . Finally,  $(x_i^*) \in \oplus^2 L_i$ .  $\square$

*Remark 2.5.* In the preliminaries of [Sch60], R. Schue wrote : “The Hilbert space direct sum of  $L^*$ -algebras defines an  $L^*$ -algebra in the obvious way”. Actually, the condition on the uniform bound of operators  $\text{ad}(x)$  is necessary.

### 3 Construction of a $L^*$ -algebra

#### 3.1 Riemannian symmetric spaces

A *Riemannian manifold* is a pair  $(M, g)$  such that  $M$  is a connected smooth manifold modeled on a real Hilbert space and  $g$  is a smooth Riemannian metric on  $M$ . Our standard reference for these manifolds is [Lan99] and in particular, we will adopt the same convention for the sign of the Riemann 4-tensor, which is also the sign used in [Hel01] for example, but is opposite to the one used in [Kli95]. With this convention, for two orthogonal unitary vectors  $u, v$  of a tangent space  $T_p M$ , the sectional curvature is  $\text{Sec}(u, v) = -R(u, v, u, v)$  where  $R$  is the Riemann 4-tensor. This convention will also explain the minus sign which appears in the definition of the curvature operator.

**Definition 3.1.** A *Riemannian symmetric space* is a Riemannian manifold such that at each point  $p \in M$ , the exponential map is surjective and there is an isometry,  $\sigma_p$  which leaves  $p$  fixed and satisfies  $d_p \sigma_p = -\text{Id}$ .

Riemannian symmetric spaces are automatically geodesically complete and metrically complete ; nevertheless, for general Riemannian manifolds of infinite dimension, metric completeness does not imply the existence of a path of minimal length between two points. So, surjectivity of the exponential map is a part of the definition.

We collect some remarks about metric completeness and geodesic completeness. In finite dimension, these two notions of completeness are equivalent thanks to Hopf-Rinow

theorem. Moreover, in finite dimension, any of this two conditions implies the existence of a path of minimal length between two points. In general, a Riemannian manifold which is metrically complete is also geodesically complete but the converse is false. Furthermore, J.H. McAlpin [Mca65] constructed a metrically complete Riemannian manifold such that there are two points which are not joined by a path of minimal length (see [Lan99, Remark p.226]).

If the sectional curvature is nonpositive then metric completeness is equivalent to geodesic completeness [Lan99, Corollary IX.3.9]. This is a consequence of a version of Cartan-Hadamard theorem due to J.H. McAlpin [Mca65]. Since any Riemannian manifold that has a symmetry at each point, is geodesically complete (see [Lan99, Proposition XIII.5.2]), this version of Cartan-Hadamard theorem [Lan99, Theorem IX.3.8] implies also that a Riemannian manifold of nonpositive sectional curvature with a symmetry at each point is a Riemannian symmetric space. Actually, in this case, the exponential map at any point is a covering, hence surjective.

### 3.2 Reminiscence of a Killing form

For the remainder of this section  $(M, g)$  will be a Riemannian symmetric space. A *Killing field* on  $M$  is a smooth vector field such that its flow is realized by isometries (metric Killing vector field in the terms of [Lan99]). Let  $\mathfrak{g}$  be the Lie algebra of Killing fields of  $M$  and let  $p$  be a point in  $M$ . The Lie algebra  $\mathfrak{g}$  has a direct decomposition  $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{p}$  where  $\mathfrak{p}$  identifies with  $T_p M$  under the map  $X \mapsto X(p)$  and  $\mathfrak{q}$  is the kernel of this map (see [Lan99, Theorem XIII.5.8]). Moreover, we have the following relations (see [Lan99, Theorem XIII.4.4])

$$\begin{aligned} [\mathfrak{q}, \mathfrak{q}] &\subseteq \mathfrak{q} \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq \mathfrak{q} \\ [\mathfrak{q}, \mathfrak{p}] &\subseteq \mathfrak{p}. \end{aligned}$$

The Riemann 4-tensor has a particular expression (see [Lan99, Theorem XIII.4.6]) in this case : for any  $X, Y, Z, T \in T_p M \simeq \mathfrak{p}$ ,

$$R(X, Y, Z, T) = g([Z, [X, Y]], T). \quad (3.1)$$

Moreover, in the particular case of a finite dimensional irreducible symmetric space, the metric on the tangent space is a multiple of the Killing form  $B$  of the group of isometries and thus

$$R(X, Y, Z, T) = \lambda B([X, Y], [Z, T]), \quad \lambda \in \mathbb{R}^*. \quad (3.2)$$

In finite or infinite dimension, the symmetries of  $R$  allows us to define a symmetric bilinear form on the alternating algebraic tensor product  $\bigwedge^2 \mathfrak{p}$  by

$$(X \wedge Y, Z \wedge T) = R(X, Y, Z, T).$$



The space  $\bigwedge^2 \mathfrak{p}$  has also a structure of preHilbert space defined by

$$\langle X \wedge Y, Z \wedge T \rangle_g = \det \begin{bmatrix} g(X, Z) & g(X, T) \\ g(Y, Z) & g(Y, T) \end{bmatrix}.$$

With these notations, the sectional curvature of two vectors  $X, Y \in T_p M$  is

$$\text{Sec}(X, Y) = -\frac{(X \wedge Y, X \wedge Y)}{\langle X \wedge Y, X \wedge Y \rangle_g}.$$

The vector space  $\bigwedge^2 \mathfrak{p}$  can be naturally identified with the space of finite rank and skew-symmetric operators of  $\mathfrak{p}$ . The tensor  $X \wedge Y = X \otimes Y - Y \otimes X$  is identified with the operator  $Z \mapsto \langle X, Z \rangle Y - \langle Y, Z \rangle X$ . This identification is actually an isometry when the space of finite rank operators is seen as a subspace of Hilbert-Schmidt operators with the Hilbert-Schmidt norm (up to a factor  $\sqrt{2}$ ). For a bounded operator  $A$  and a finite rank operator  $B$  on  $\mathfrak{p}$ , we define

$$\langle A, B \rangle_g = \text{trace}({}^t AB).$$

For example, if  $A$  is a bounded operator and  $X, Y \in \mathfrak{p}$  then  $\langle A, X \wedge Y \rangle_g = g(AX, Y) - g(X, AY)$ .

In finite dimension (see, e.g., [Pet06, Section 2.2] or [GM75, §4]),  $\langle, \rangle_g$  is simply the Hilbert-Schmidt scalar product on  $L(\mathfrak{p})$  (where  $L(\mathfrak{p})$  is the space of linear bounded operators on  $\mathfrak{p}$ ) and thus there is a symmetric operator  $C$  of  $\bigwedge^2 \mathfrak{p}$  such that

$$(X \wedge Y, Z \wedge T) = -\langle C(X \wedge Y), Z \wedge T \rangle_g$$

for  $X, Y, Z, T \in \mathfrak{p}$ . This operator is called the *curvature operator* of  $M$ .

In infinite dimension, we can generalize this construction by defining a similar operator  $C: \bigwedge^2 \mathfrak{p} \rightarrow L(\mathfrak{p})$  such that  $(X \wedge Y, Z \wedge T) = -\langle C(X \wedge Y), Z \wedge T \rangle_g$ . Actually,  $C(X \wedge Y)$  is skew-symmetric and thanks to equation (3.1), we know that  $C(X \wedge Y)Z = 1/2[Z, [X, Y]]$ . We call  $C$  the *curvature operator* of  $M$ .

We say that the curvature operator is *nonpositive* (respectively *nonnegative*) if for any  $U \in \bigwedge^2(\mathfrak{p})$ ,  $\langle C(U), U \rangle_g \leq 0$  (respectively  $\langle C(U), U \rangle_g \geq 0$ ). Observe that  $C$  is nonpositive (respectively nonnegative) if for any families  $(X_i)_{i=1\dots n}$ ,  $(Y_i)_{i=1\dots n}$ ,

$$\sum_{i,j=1}^n R(X_i, Y_i, X_j, Y_j) \geq 0$$

(respectively  $\sum_{i,j} R(X_i, Y_i, X_j, Y_j) \leq 0$ ).

Now we assume that  $(M, g)$  is a Riemannian symmetric space of fixed-sign curvature operator. For brevity, we will write  $M$  is NPCO (resp. NNCO) if  $M$  has nonpositive

curvature operator (resp. nonnegative curvature operator). We want to endow  $[\mathfrak{p}, \mathfrak{p}]$  with a structure of preHilbert space. For  $U = \sum_i [X_i, Y_i]$  and  $V = \sum_j [Z_j, T_j]$ , we define  $\langle U, V \rangle = -\sum_j g([U, Z_j], T_j)$  if  $M$  is NPCO and  $\langle U, V \rangle = \sum_j g([U, Z_j], T_j)$  if  $M$  is NNCO. For example, if  $M$  is NPCO

$$\langle U, V \rangle = \sum_{i,j} R(X_i, Y_i, Z_j, T_j) = \sum_{i,j} (X_i \wedge Y_i, Z_j \wedge T_j).$$

The symmetries of the Riemann tensor imply this is a symmetric bilinear form and the hypothesis on the curvature operator implies this form is nonnegative in both cases. The relation  $R(X, Y, Z, T) = R(Z, T, X, Y)$  for  $X, Y, Z, T \in \mathfrak{p}$  implies for any  $U \in [\mathfrak{p}, \mathfrak{p}]$  that

$$g([X, U], Y) = \langle U, [X, Y] \rangle \quad (3.3)$$

if  $M$  is NPCO and

$$g([X, U], Y) = -\langle U, [X, Y] \rangle \quad (3.4)$$

if  $M$  is NNCO. Moreover, the Cauchy-Schwarz inequality implies that if  $\langle U, U \rangle = 0$  then for any  $X, Y \in \mathfrak{p}$ ,  $g([U, X], Y) = \pm \langle U, [X, Y] \rangle = 0$  and thus the Killing field  $U$  is trivial. We denote by  $\mathfrak{k}$  the completion of  $[\mathfrak{p}, \mathfrak{p}]$  with respect to  $\langle \cdot, \cdot \rangle$  and we extend  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{k} \oplus \mathfrak{p}$  such that  $\mathfrak{p}$  and  $\mathfrak{k}$  are orthogonal and the restriction of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$  coincides with  $g$ .

*Proof of Theorem 1.2.* We show that the Lie algebra structure on  $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$  extends to a  $L^*$ -algebra structure on  $\mathfrak{k} \oplus \mathfrak{p}$ . Since the Riemann 4-tensor is a bounded 4-linear form at each point, there exists a constant  $\kappa$  such that  $R(X, Y, Z, T) \leq \kappa \|X\| \|Y\| \|Z\| \|T\|$  for any  $X, Y, Z, T \in \mathfrak{p}$ . Thus  $\|[X, Y]\| \leq \sqrt{\kappa} \|X\| \|Y\|$ . If  $U \in \mathfrak{k}$  and  $X, Y \in \mathfrak{p}$  then  $|\langle X, [V, Y] \rangle| = |\langle V, [X, Y] \rangle| \leq \|V\| \cdot \|[X, Y]\|$ . The Lie bracket extends continuously to  $\mathfrak{k} \times \mathfrak{p}$  and any  $U \in \mathfrak{k}$  defines a bounded skew-symmetric operator  $X \mapsto [U, X]$ .

Moreover, Jacobi's identity for  $U \in [\mathfrak{p}, \mathfrak{p}]$  and  $X, Y \in \mathfrak{p}$ ,

$$[U, [X, Y]] = [[U, X], Y] + [X, [U, Y]],$$

shows that  $[\mathfrak{p}, \mathfrak{p}]$  is a subalgebra of the algebra of Killing fields. For  $X, Y \in \mathfrak{p}$  and  $U, V \in [\mathfrak{p}, \mathfrak{p}]$  we have

$$\begin{aligned} |\langle [U, V], [X, Y] \rangle| &= |\langle [X, [U, V]], Y \rangle| \\ &= |\langle [[X, U], V] + [U, [X, V]], Y \rangle| \\ &= |\langle [X, U], [V, Y] \rangle - \langle [X, V], [U, Y] \rangle| \\ &\leq 2\sqrt{\kappa} \|U\| \|V\| \|X\| \|Y\|. \end{aligned}$$

This shows that the map  $U, V \mapsto [U, V]$  extends continuously to  $\mathfrak{k} \times \mathfrak{k}$  (endowed with the product topology of the strong topology) when the target  $\mathfrak{k}$  is endowed with the weak topology.

We now define the involution. For  $U \in \mathfrak{k}$ , we set  $U^* = -U$  and for  $X \in \mathfrak{p}$ , we set  $X^* = X$  if the curvature operator is nonpositive and  $X^* = -X$  if the curvature operator is nonnegative. It remains to show that

$$\langle [X, Y], Z \rangle = \langle Y, [X^*, Z] \rangle \quad (3.5)$$

for any  $X, Y, Z \in \mathfrak{k} \oplus \mathfrak{p}$ . Thanks to linearity and relations  $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$ ,  $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ ,  $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$  and  $\mathfrak{k} \perp \mathfrak{p}$ , it suffices to show Equation (3.5) in the case  $X \in \mathfrak{k}$ ,  $Y, Z \in \mathfrak{p}$  and in the case  $X, Y, Z \in \mathfrak{k}$ . Suppose that  $X \in \mathfrak{k}$ ,  $Y, Z \in \mathfrak{p}$  then using Equations (3.3) and (3.4) we have

$$\langle [X, Y], Z \rangle = \pm \langle X, [Z, Y] \rangle = \mp \langle X, [Y, Z] \rangle = \mp \langle [X, Z], Y \rangle = \langle Y, [X^*, Z] \rangle.$$

For the case  $X, Y, Z \in \mathfrak{k}$ , thanks to continuity and linearity, we assume that  $X = [X_1, X_2]$  for some  $X_1, X_2 \in \mathfrak{p}$ . We treat only the case where  $M$  is NPCO, the other case is similar.

$$\begin{aligned} \langle [X, Y], Z \rangle &= \langle [[X_1, X_2], Y], Z \rangle \\ &= - \langle [[Y, X_1], X_2] + [X_1, [Y, X_2]], Z \rangle \\ &= - \langle [Y, X_1], [Z, X_2] \rangle - \langle [Y, X_2], [X_1, Z] \rangle \\ &= - \langle Y, [[Z, X_2], X_1] + [[X_1, Z], X_2] \rangle \\ &= \langle Y, [Z, [X_1, X_2]] \rangle \\ &= - \langle Y, [X, Z] \rangle = \langle Y, [X^*, Z] \rangle. \end{aligned}$$

□

**Proposition 3.2.** *Let  $(M, g)$  be a Riemannian symmetric space and let  $L, L'$  be  $L^*$ -algebras with orthogonal decompositions  $L = \mathfrak{k} \oplus \mathfrak{p}$  and  $L' = \mathfrak{k}' \oplus \mathfrak{p}'$  satisfying (i) and (ii) of Theorem 1.2 then  $L$  and  $L'$  are isomorphic.*

*Proof.* First,  $\mathfrak{p}$  and  $\mathfrak{p}'$  are isometric as Hilbert spaces and they generate isomorphic Lie algebras. Now, it suffices to observe that this isomorphism is also an isometry since the inner products are determined by their respective restrictions on  $\mathfrak{p}$  and  $\mathfrak{p}'$ . □

We state a little bit more precise theorem than Theorem 1.3.

**Theorem 3.3.** *Let  $(M, g)$  and  $(M', g')$  be Riemannian symmetric spaces with points  $p \in M$  and  $p' \in M'$ . Let  $L, L'$  be two  $L^*$ -algebras with orthogonal decompositions  $L = \mathfrak{k} \oplus \mathfrak{p}$  and  $L' = \mathfrak{k}' \oplus \mathfrak{p}'$  satisfying properties (i) and (ii) of Theorem 1.2 with respect to  $p \in M$  and  $p' \in M'$ .*

*Assume there exists an isomorphism of  $L^*$ -algebras between  $L$  and  $L'$  which intertwines the previous orthogonal decompositions. If  $B(p, r)$ ,  $B(p', r)$  are normal neighborhoods then  $B(p, r)$  and  $B(p', r)$  are isometric.*

*Proof.* The isomorphism will be provided by Cartan's theorem [Kli95, Theorem 1.12.8]. Let  $\varphi$  be an isomorphism between  $L$  and  $L'$  such that  $\varphi(\mathfrak{k}) = \mathfrak{k}'$  and  $\varphi(\mathfrak{p}) = \mathfrak{p}'$ . We define  $i_p: T_p M \rightarrow T_{p'} M'$  to be the restriction of  $\varphi$  to  $\mathfrak{p}$  identified with  $T_p M$ . The map  $i_p$  is a linear isometry between Hilbert spaces. We define  $\Phi = \exp_{p'} \circ i_p \circ \exp_p^{-1}: B(p, r) \rightarrow B(p', r)$ .

First, since  $\varphi$  is a Lie algebra isomorphism and an isometry

$$R'(\varphi(X), \varphi(Y), \varphi(Z), \varphi(T)) = < [\varphi(Z), [\varphi(X), \varphi(Y)]], \varphi(T) > = R(X, Y, Z, T)$$

for any  $X, Y, Z, T \in T_p M$ . For any Riemannian manifold  $N$  with Riemannian 4-tensor  $R$ , a point  $q \in N$  and  $X \in T_q N$ , we denote by  $R_X: T_q N \rightarrow T_q N$  the symmetric operator such that  $R_X(Y) = R(X, Y)X = [X, [X, Y]]$  for any  $Y \in T_q N$ .

If  $c$  is a geodesic curve  $c: [a, b] \rightarrow M$  we denote by  $\dot{c}(t)$  the tangent vector at  $c(t)$  and  $P_{a,c}^b$  the parallel transport along  $c$ . It is shown in [Lan99, XIII, §6] that the Riemann tensor of a Riemannian symmetric space is parallel :

$$P_{a,c}^b \circ R_{\dot{c}(a)} = R_{\dot{c}(b)} \circ P_{a,c}^b.$$

Now, let  $c$  be a radial geodesic with unit speed starting at  $p$  and let  $c'$  be its image by  $\Phi$ . For  $0 \leq t < r$  we set  $\mathbf{i}_t = P_{0,c'}^t \circ \mathbf{i}_p \circ P_{t,c}^0$ . Hence,

$$\begin{aligned} \mathbf{i}_t \circ R_{\dot{c}(t)} &= P_{0,c'}^t \circ \mathbf{i}_p \circ P_{t,c}^0 \circ R_{\dot{c}(t)} \\ &= P_{0,c'}^t \circ \mathbf{i}_p \circ R_{\dot{c}(0)} \circ P_{t,c}^0 \\ &= P_{0,c'}^t \circ R_{\dot{c}'(0)} \circ \mathbf{i}_p \circ P_{t,c}^0 \\ &= R_{\dot{c}'(t)} \circ \mathbf{i}_t. \end{aligned}$$

The hypotheses of Cartan's theorem are now satisfied.  $\square$

The following proposition gives a natural condition which implies a decomposition as asked in Question 1.4.

**Proposition 3.4.** *Let  $M$  be a Riemannian symmetric space. If there exists a dense increasing union of totally geodesic subspaces of finite dimension containing a point  $p \in M$ , then there is an orthogonal decomposition*

$$T_p M = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+$$

such that

- the subspaces  $\mathfrak{p}_-$ ,  $\mathfrak{p}_0$  and  $\mathfrak{p}_+$  are commuting Lie triple systems of the Lie algebra of Killing fields,
- the restrictions of the curvature operator are nonnegative on  $\mathfrak{p}_-$ , trivial on  $\mathfrak{p}_0$  and nonpositive on  $\mathfrak{p}_+$ .

*Proof.* Let  $(M_n)$  be an increasing sequence of finite dimensional totally geodesic subspaces of  $M$  such that their union is dense in  $M$ . Choose  $p \in M_1$  and let  $R^{M_n}$  be the Riemannian tensor of  $M_n$  at  $p$ . Since  $M_n$  is totally geodesic in  $M$ , for any  $X, Y, Z, T \in T_p M_n$ ,  $R^{M_n}(X, Y, Z, T) = R(X, Y, Z, T)$  (see [Lan99, Corollary XIV.1.4]). Moreover, for any  $x \in M_n$ ,  $\sigma_x(M_n) = M_n$  and thus  $M_n$  is a Riemannian symmetric space

on its own. Now, The tangent space  $\mathfrak{p}_n := T_p M_n$  can be decomposed as  $\mathfrak{p}_n^n \oplus \mathfrak{p}_0^n \oplus \mathfrak{p}_+^n$  where  $\mathfrak{p}_n^n$ ,  $\mathfrak{p}_0^n$  and  $\mathfrak{p}_+^n$  satisfy properties of the proposition. We claim that for  $m > n$ ,  $\mathfrak{p}_n^n \subseteq \mathfrak{p}_m^m$  and  $\mathfrak{p}_+^n \subseteq \mathfrak{p}_+^m$ . Actually, if  $\mathfrak{g}^n$  is the Lie subalgebra  $[\mathfrak{p}_n, \mathfrak{p}_n] \oplus \mathfrak{p}_n$  of the isometry group of  $M_n$  then it is an orthogonal symmetric Lie algebra (see [Hel01, Chapters IV and V]) which can be decomposed as

$$\mathfrak{g}^n = \mathfrak{g}_-^n \oplus \mathfrak{p}_0^n \oplus \mathfrak{g}_+^n$$

where  $\mathfrak{g}_-^n, \mathfrak{g}_+^n$  are respectively of compact and noncompact types and  $\mathfrak{p}_0^n$  is the maximal central Abelian subspace of  $\mathfrak{p}_n$ . In particular,  $\mathfrak{g}^n$  is a subalgebra of  $\mathfrak{g}^m$  and  $\mathfrak{s}_n := \mathfrak{g}_-^n \oplus \mathfrak{g}_+^n$  is a semisimple Lie algebra and thus contained in  $\mathfrak{s}_m$ . The semisimple algebras  $\mathfrak{s}_n$  and  $\mathfrak{s}_m$  are orthogonal sums of simple ideals of compact or noncompact types. Let  $\pi$  be the orthogonal projection on a simple ideal  $J$  of  $\mathfrak{s}_m$ . The restriction of  $\pi$  to any simple ideal  $I$  of  $\mathfrak{s}_n$  is either trivial or is an isomorphism of orthogonal symmetric Lie algebras on its image. In particular, if  $\pi(I) \neq \{0\}$  then  $I$  and  $J$  have same type (compact or noncompact). This proves the claim.

We set  $\mathfrak{p}_+ = \overline{\cup_n \mathfrak{p}_+^n}$ ,  $\mathfrak{p}_- = \overline{\cup_n \mathfrak{p}_-^n}$  and  $\mathfrak{p}^0 = \{X \in \mathfrak{p}, [X, Y] = 0, \forall Y \in \mathfrak{p}\}$ . Let  $X \in (\mathfrak{p}_+ \oplus \mathfrak{p}_-)^{\perp}$ , then if  $\pi_n: \mathfrak{p} \rightarrow \mathfrak{p}_n$  is the orthogonal projection on  $\mathfrak{p}_n$  then  $\pi_n(X) \in \mathfrak{p}_0^n$ . Actually for any  $Y \in \mathfrak{p}$ ,

$$\begin{aligned} [Y, X] = 0 &\iff [Z, [X, Y]] = 0, \forall Z \in \mathfrak{p} \\ &\iff g([Z, [X, Y]], T) = R(X, Y, Z, T) = 0, \forall Z, T \in \mathfrak{p}. \end{aligned}$$

Thus,  $R(X, Y, Z, T) = \lim_n R(\pi_n(X), \pi_n(Y), Y, T) = 0$  for any  $Z, T \in \mathfrak{p}$  and  $[X, Y] = 0$ . Therefore  $(\mathfrak{p}_+ \oplus \mathfrak{p}_-)^{\perp} \subseteq \mathfrak{p}^0$ . If we set  $\mathfrak{p}_0 = (\mathfrak{p}_+ \oplus \mathfrak{p}_-)^{\perp}$  then we have the desired decomposition.  $\square$

## 4 Nonpositive curvature

### 4.1 Geometry of nonpositive curvature

A Riemannian manifold of finite dimension is locally CAT(0) (or is nonpositively curved in the sense of Alexandrov) if and only if it has nonpositive sectional curvature. The same result is also true in infinite dimension and a proof can be found in [Lan99, Theorem IX.3.5]. We refer to [BH99] for generalities about CAT(0) spaces.

**Proposition 4.1.** *If  $(M, g)$  is a Riemannian symmetric space of noncompact type then  $M$  is simply connected, the exponential map at any point is a diffeomorphism and  $M$  is CAT(0).*

*Proof.* Assume  $(M, g)$  is a Riemannian symmetric space of noncompact type and consider its universal covering  $\widetilde{M}$ . This universal covering has a natural structure of Riemannian manifold turning the projection  $\pi: M \rightarrow \widetilde{M}$  into a Riemannian covering. In that way  $\widetilde{M}$  is simply connected and is locally CAT(0) since  $M$  is locally CAT(0). The space  $\widetilde{M}$

is a CAT(0) space thanks to Cartan-Hadamard theorem [BH99, Theorem II.4.1].

Choose  $\tilde{x}, \tilde{y} \in \widetilde{M}$ . The projection of the geodesic segment between  $\tilde{x}$  and  $\tilde{y}$  is a (locally minimizing) geodesic segment between  $x = \pi(\tilde{x})$  and  $y = \pi(\tilde{y})$ . Let  $f_t$  be the isometry  $\sigma_{x_t} \circ \sigma_x$  where  $x_t$  is the point at distance  $td(\tilde{x}, \tilde{y})/2$  from  $x$  on the previous segment and  $t \in [0, 1]$ . Let  $(F_t)_{t \in [0, 1]}$  be a lift of  $(f_t)_{t \in [0, 1]}$  such that  $F_0 = \text{Id}$ . Remark that  $t \mapsto F_t(\tilde{x})$  is a lift of the geodesic segment from  $x$  to  $y$  and since  $F_0(\tilde{x}) = \tilde{x}$ , this is the geodesic from  $\tilde{x}$  to  $\tilde{y}$  and thus  $F_1(\tilde{x}) = \tilde{y}$ . Since  $\pi$  is a Riemannian covering, we observe that  $F_t$  is an isometry of  $\widetilde{M}$  for any  $t \in [0, 1]$ .

For  $\gamma \in \pi_1(M)$  and  $t \in [0, 1]$ ,

$$\pi \circ F_t \circ \gamma = f_t \circ \pi \circ \gamma = f_t \circ \pi = \pi \circ F_t.$$

The map  $\pi \circ F_t$  is a Riemannian covering and thus for any  $t$ , there exists  $\gamma'$  such that  $F_t \circ \gamma = \gamma' \circ F_t$ . A connectedness argument shows that  $\gamma'$  is independent of  $t$  and since  $F_0 = \text{Id}$  then  $\gamma' = \gamma$ . This shows that the displacement function of  $\gamma$  is the same at  $x$  and at  $y$  and thus is constant on  $\widetilde{M}$ . Suppose this displacement length is not zero then  $\gamma$  is a Clifford translation,  $\widetilde{M}$  has a Euclidean factor and  $\widetilde{M} \simeq \mathbb{R} \times \widetilde{N}$  as metric space. Now let  $X$  be the unit vector field pointing in the direction of the Euclidean factor. The vector field  $X$  is a Killing field and  $\pi_* X$  is also Killing since Killing fields are characterized by a differential equation. This Killing field has a trivial Lie bracket with any other Killing vector field. This is a contradiction with the hypothesis of noncompact type and thus  $\gamma$  is trivial.

Since we know that  $M$  is simply connected, Cartan-Hadamard theorem [Lan99, Theorem IX.3.8] shows that the exponential map at any point is a diffeomorphism.  $\square$

## 4.2 $L^*$ -algebras of noncompact type

For the remainder of the section,  $(M, g)$  will be a separable Riemannian symmetric of noncompact type.

**Lemma 4.2.** *The  $L^*$ -algebra associated to  $M$  is a Hilbertian sum  $L = \oplus L_i$  of simple  $L^*$ -algebras  $L_i$  of noncompact type.*

*Proof.* Since  $L$  has no nontrivial abelian ideal then  $L$  is semisimple and can be written uniquely  $L = \oplus L_i$  where each  $L_i$  is simple. Assume for contradiction that there is a  $L_i$  which is of compact type. By construction  $L = \overline{[\mathfrak{p}, \mathfrak{p}]} \oplus \mathfrak{p}$  and since  $L_i$  is invariant under  $*$  then  $L_i \subset \overline{[\mathfrak{p}, \mathfrak{p}]}$ . Thus,  $\mathfrak{p} \subseteq \oplus_{j \neq i} L_j$ ,  $[\mathfrak{p}, L_i] = 0$  and  $[L_i, L] = 0$ , which is a contradiction.  $\square$

Thanks to the classification, we know that each  $L_i$  that has infinite dimension, is homothetic to one element of the following list.

Type	Algebra
A I	$\mathfrak{gl}_\infty^2(\mathbb{R})$
A II	$\mathfrak{u}_\infty^{*2}(\mathbb{C})$
A III	$\mathfrak{u}^2(p, \infty), p \in \mathbb{N}^* \cup \{\infty\}$
BD I	$\mathfrak{o}^2(p, \infty), p \in \mathbb{N}^* \cup \{\infty\}$
BD III	$\mathfrak{o}^{*2}(\infty)$
C I	$\mathfrak{sp}_\infty^2(\mathbb{R})$
C II	$\mathfrak{sp}^2(p, \infty), p \in \mathbb{N}^* \cup \{\infty\}$
A	$\mathfrak{gl}_\infty^2(\mathbb{C})$
BD	$\mathfrak{o}_\infty^2(\mathbb{C})$
C	$\mathfrak{sp}_\infty^2(\mathbb{C})$

The last three algebras are moreover complex simple  $L^*$ -algebras. The notations used here are maybe not standard but we hope the correspondence with notations used in [dlH71] or [Uns71] is transparent. They are chosen to be brief and close to the ones used in finite dimension [Hel01, Tables IV and V, X.6]. We refer to one of the previous references for a description of these algebras.

Each of these algebras can be realized as a  $L^*$ -subalgebra of  $\mathfrak{gl}_\infty^2(\mathbb{R})$ , which is the Lie algebra of Hilbert-Schmidt operators of some real separable Hilbert space  $\mathcal{H}$ , endowed with the Hilbert-Schmidt norm. For  $X \in \mathfrak{gl}_\infty^2(\mathbb{R})$ ,  $X^*$  is the adjoint of  $X$  as operator on  $\mathcal{H}$ . The algebra  $\mathfrak{gl}_\infty^2(\mathbb{R})$  is the Lie algebra of the Hilbert-Lie group  $\mathrm{GL}_\infty^2(\mathbb{R})$ . If  $\mathrm{O}^2(\infty)$  is the intersection of  $\mathrm{GL}_\infty^2(\mathbb{R})$  and the orthogonal group  $\mathrm{O}(\mathcal{H})$  of  $\mathcal{H}$  then  $\mathrm{GL}_\infty^2(\mathbb{R})/\mathrm{O}^2(\infty)$  is a Riemannian symmetric space of noncompact type (see for example [dlH72, III.2]).

Let  $\mathfrak{g}$  be any  $L^*$ -algebra of the previous list viewed as a  $L^*$ -subalgebra of  $\mathfrak{gl}_\infty^2(\mathbb{R})$ . Let  $G$  be the closed subgroup of  $\mathrm{GL}_\infty^2(\mathbb{R})$  generated by  $\exp \mathfrak{g}$  and  $K = G \cap \mathrm{O}(\mathcal{H})$ . If  $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$  is the decomposition of  $\mathfrak{g}$  into skew-symmetric and symmetric parts then thanks to [dlH72, Proposition III.4],  $\exp(\mathfrak{p})$  is a totally geodesic subspace of  $\mathrm{GL}_\infty^2(\mathbb{R})/\mathrm{O}^2(\infty)$ ,  $G$  acts transitively on  $\exp(\mathfrak{p})$  and  $K$  is the stabilizer of  $\mathrm{Id}$  in  $G$ . In this way,  $\exp(\mathfrak{p}) \simeq G/K$ . When  $\mathfrak{g}$  varies among the elements of the previous list, one obtains the irreducible symmetric spaces of noncompact type which appear in Theorem 1.7.

If  $L$  is a simple  $L^*$ -algebra of noncompact, let  $\mathfrak{g}$  be the element of the homothety class of  $L$  that is in the previous list and let  $\lambda$  be the scaling factor such that  $L = \lambda \cdot \mathfrak{g}$ . The *Riemannian symmetric space associated to  $L$*  is the space  $G/K$  endowed with the metric that is the multiple by  $\lambda$  of the metric coming from the embedding in  $\mathrm{GL}_\infty^2(\mathbb{R})/\mathrm{O}^2(\infty)$ .

It is a routine verification to show that if one starts from a simple  $L^*$ -algebra of noncompact type  $L$ , one considers the Riemannian symmetric space  $M$  associated to  $L$  and one constructs the  $L^*$ -algebra as in Section 3.2 then the  $L^*$ -algebra constructed is isomorphic to  $L$ .

*Remark 4.3.* If  $L$  is a simple  $L^*$ -algebra of noncompact type and of finite dimension then it is a simple Lie algebra of noncompact type in the usual sense. It is associated to a

Riemannian symmetric space of noncompact type (in the usual sense) and  $L$  coincides with the  $L^*$ -algebra associated to this Riemannian symmetric space. See Example 2.3 and [Hel01, Chapter V]. Moreover,  $L$  embeds (up to homothety) as a  $L^*$ -subalgebra of  $\mathrm{SL}_n(\mathbb{R})$  for some  $n$ .

**Proposition 4.4.** *Let  $M$  be a simply connected Riemannian symmetric space with fixed-sign curvature operator. Let  $L = \mathfrak{k} \oplus \mathfrak{p}$  be the  $L^*$ -algebra associated to  $M$  at a point  $p$ . If  $I$  is an ideal of  $L$  invariant under  $d_p\sigma_p$  then  $N = \exp_p(I \cap \mathfrak{p})$  is a totally geodesic subspace of  $M$ .*

*Proof.* Let  $q$  be an other point of  $M$ . One can also associate a  $L^*$ -algebra  $L_q$  with respect to  $q$ . If  $\gamma$  is Riemannian isometry such that  $\gamma p = q$  then the differential of  $\gamma$  induces an isomorphism between  $L$  and  $L_q$ . In particular, the image of an ideal is also an ideal. Let  $E = I \cap \mathfrak{p}$  and for  $q \in M$  let  $E_q$  be the parallel transport of  $E$  along the geodesic segment from  $p$  to  $q$  (this way  $E = E_p$ ). Since this parallel transport is realized by the differential of the transvection from  $p$  to  $q$  then  $E_q$  is the intersection of an ideal  $I_q$  of  $L_q$  and  $\mathfrak{p}_q$  (where  $L_q = \mathfrak{k}_q \oplus \mathfrak{p}_q$  is the decomposition obtained in Theorem 1.2 with respect to  $q$ ). Observe that if  $q, q'$  are two points in  $M$  then the parallel transport of  $E_q$  along the geodesic segment from  $q$  to  $q'$  is  $E_{q'}$ . Actually the composition of the differentials of the transvections from  $p$  to  $q$ , from  $q$  to  $q'$  and from  $q'$  to  $p$  maps  $I$  to an ideal  $I'$  which depends continuously on  $q$  and  $q'$  and thus is  $I$  since the geodesic loop  $p \rightarrow q \rightarrow q' \rightarrow p$  can be contracted continuously (along geodesic segments  $[p, q]$  and  $[p, q']$ ) to the constant loop at  $p$ .

In the terminology of the theorem of Frobenius [Lan99, Theorem VI.1.1],  $(E_q)$  is a tangent subbundle which is integrable. Any maximal integrable manifold of  $(E_q)$  is totally geodesic thanks to the same argument which appears at the second page of [dR52]. In particular, the maximal integral manifold containing  $p$  is  $\exp_p(E)$  and is totally geodesic.  $\square$

Before proving Theorem 1.7, we make the following observation. In the situation of Proposition 4.4, the totally geodesic submanifold  $N = \exp_p(I \cap \mathfrak{p})$  is invariant under symmetry of  $M$  at  $q \in N$  and thus is a Riemannian symmetric space on its own. Any Killing field on  $N$  is the restriction of a Killing field on  $M$ . This shows that the  $L^*$ -algebra associated to  $N$  is  $I$ .

*Proof of Theorem 1.7.* Let  $M$  be a symmetric space of noncompact type and  $L = \mathfrak{k} \oplus \mathfrak{p}$  be its associated  $L^*$ -algebra. This algebra  $L$  is a Hilbertian sum  $L = \oplus^2 L_i$  of simple  $L^*$ -algebras of noncompact type. We set  $N_i$  to be the totally geodesic submanifold of  $M$  constructed from  $L_i$  as in Proposition 4.4 and  $N'_n$  the one associated to the ideal  $L'_n := \oplus_{i>n}^2 L_i$ . The Riemannian product  $N_1 \times \cdots \times N_n \times N'_n$  is a Riemannian symmetric space with associated  $L^*$ -algebra  $L_1 \oplus \cdots \oplus L_n \oplus L' \simeq L$ . Thanks to Theorem 3.3, we know that  $M$  and  $N_1 \times \cdots \times N_n \times N'_n$  are diffeomorphically isometric.



Now, let  $M'$  be the Hilbertian product  $\prod_i^2 N_i$  and  $\varphi: \mathfrak{p} \rightarrow M'$  the homeomorphism defined by  $\varphi((X_i)) = (\exp_p(X_i))$ . The map  $\exp_p \circ \varphi^{-1}$  is an homeomorphism from  $M'$  to  $M$  which is isometric on the dense subset  $\{(\exp_p(X_i)) \mid X_i \neq 0 \text{ for finitely many } i\}$ . Thus  $\exp_p \circ \varphi^{-1}$  is an isometry.  $\square$

*Remark 4.5.* Let  $X = \prod_{i \in I}^2 X_i$  and  $Y = \prod_{j \in J}^2 Y_j$  be two Hilbertian products of pointed metric spaces  $(X_i, x_i, d_i)$  and  $(Y_j, y_j, \delta_j)$ . We say that  $X$  and  $Y$  are *multihomothetic* if there exists a bijection  $\varphi: I \rightarrow J$ , a family of scaling factors  $(\lambda_i)_{i \in I}$  and isometries  $\Phi_i: (X_i, \lambda_i d_i) \rightarrow (Y_{\varphi(i)}, \delta_{\varphi(i)})$  such that  $\Phi_i(x_i) = y_{\varphi(i)}$ .

We emphasize that the diagonal map between cartesian products

$$\begin{aligned} \Phi: \prod X_i &\rightarrow \prod Y_j \\ (x_i) &\mapsto (\Phi_{\varphi^{-1}(j)}(x_{\varphi^{-1}(j)})) \end{aligned}$$

induces a bijection, which is a homeomorphism, between  $X$  and  $Y$  if and only if there are two positive numbers  $c, C > 0$  such that  $c \leq \lambda_i \leq C$  for all  $i \in I$ .

It is a classical fact that any Riemannian symmetric space of noncompact and finite dimension is multihomothetic to a totally geodesic subspace of  $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$  for some  $n$ . This is also true in general. Let  $M = \prod^2 M_i$  be a separable Riemannian symmetric space of non-compact type and  $L = \oplus^2 L_i$  be its associated  $L^*$ -algebra. Let  $\mathfrak{g}_i$  be the  $L^*$ -algebra homothetic to  $L_i$  that is a  $L^*$ -subalgebra of  $\mathfrak{gl}^2(\mathcal{H}_i)$  where  $\mathcal{H}_i$  is a real Hilbert space of finite or infinite dimension and  $\mathfrak{gl}^2(\mathcal{H}_i)$  is the  $L^*$ -algebra of Hilbert-Schmidt operators on  $\mathcal{H}_i$ . Let  $\mathcal{H}$  be the Hilbertian sum  $\oplus^2 \mathcal{H}_i$ . Thus,

$$\oplus^2 \mathfrak{g}_i \leq \oplus^2 \mathfrak{gl}^2(\mathcal{H}_i) \leq \mathfrak{gl}^2(\mathcal{H}).$$

The image by the exponential map of the symmetric part of  $\oplus^2 \mathfrak{g}_i$  is a totally geodesic subspace of  $\mathrm{GL}^2(\mathcal{H})/\mathrm{O}^2(\mathcal{H})$  and this space is multihomothetic to  $M$  (but the multihomothety is not necessarily a homeomorphism).

*Proof of Corollary 1.8.* If  $M$  is a separable symmetric space of noncompact type and finite rank then its associated  $L^*$ -algebra  $L$  is a finite sum of simple  $L^*$ -algebras  $L = \oplus^2 L_i$  and each  $L_i$  has finite rank. Thus  $L_i$  has finite dimension or is homothetic to  $\mathfrak{u}^2(p, \infty)$ ,  $\mathfrak{o}^2(p, \infty)$  or  $\mathfrak{sp}^2(p, \infty)$  with  $p \in \mathbb{N}$ .

The telescopic dimension is always greater or equal to the rank and it is exactly equal to the rank when the symmetric space has finite dimension or is  $X_p(\mathbb{K})$  because in both cases, any asymptotic cone is a Euclidean building of dimension equal to the rank (see [KL97] and [Duc12, Corollary 1.4]).  $\square$

### 4.3 A CAT(0) symmetric space which is not a Riemannian manifold

We describe an example of a CAT(0) symmetric space which is not a Riemannian manifold.

Let  $\mathbb{H}$  be the hyperbolic plane with constant sectional curvature  $-1$ . We fix an origin  $o \in \mathbb{H}$ . We consider  $X = L^2([0, 1], \mathbb{H})$ , the space of measurable maps  $x: t \mapsto x_t$  from  $[0, 1]$  (endowed with the Lebesgue measure) to  $\mathbb{H}$  such that  $t \mapsto d(o, x_t)$  is a square integrable function. This space (called *Pythagorean integral* in [Mon06]) endowed with the distance

$$d(x, y) = \left( \int_{[0,1]} d(x_t, y_t)^2 dt \right)^{1/2}$$

is a complete separable CAT(0) space. Geodesics can be easily described as in *loc. cit.*. Actually, if  $I$  is a real interval, a map  $g: I \rightarrow X$  is a geodesic then there exist a measurable map  $\alpha: [0, 1] \rightarrow \mathbb{R}^+$  and a collection of geodesics  $g_t: \alpha(t)I \rightarrow \mathbb{H}$  such that

$$\int_{[0,1]} \alpha(t)^2 dt = 1, \quad (g(s))_t = g_t(\alpha(t)s)$$

for all  $s \in I$  and almost all  $t \in [0, 1]$ . For  $h \in \mathbb{H}$ , let  $S_h$  be the geodesic symmetry at  $h$  in  $\mathbb{H}$ . For  $x, y \in X$ , we set  $\sigma_x(y)$  to be the map  $t \mapsto S_{x_t}(y_t)$ . The description of geodesics implies that  $S_x$  is the geodesic symmetry at  $x$ . Therefore  $X$  is a CAT(0) symmetric space.

Let  $X$  be a CAT(0) space and  $x$  be a point in  $X$ . The space of directions  $\Sigma_x$  of  $X$  at  $x$  is the set of classes of geodesic rays starting at  $x$ . Two rays are identified if their Alexandrov angle vanishes. The Alexandrov angle gives a metric on the quotient. The tangent cone  $T_x$  is the Euclidean cone over  $\Sigma_x$ . We describe  $\Sigma_x$  and  $T_x$  for  $x \in L^2([0, 1], \mathbb{H})$  below. We denote by  $\bar{\angle}_x(y, z)$  the comparison angle and by  $\angle_x(y, z)$  the Alexandrov angle at  $x$  between  $y$  and  $z$ .

**Definition 4.6.** Let  $(Y, d)$  be a separable metric space of diameter less than  $\pi$  and  $(\Omega, \mu)$  a standard measure space. The *integral join*,  $\int_{\Omega}^*$ , is the set of pairs  $(y, v) = ((y_{\omega}), (v_{\omega}))$  such that

- (i) for all  $\omega \in \Omega$ ,  $y_{\omega} \in Y$  and  $v_{\omega} \in \mathbb{R}^+$ ,
- (ii) the map  $\omega \mapsto v_{\omega}$  is measurable and  $\int_{\Omega} v_{\omega}^2 d\mu(\omega) = 1$ ,
- (iii) the map  $\omega \mapsto y_{\omega}$  is measurable.

The metric on  $\int_{\Omega}^*$  is defined by the formula

$$\cos(d((x, v), (y, w))) = \int_{\Omega} v_{\omega} w_{\omega} \cos(d(x_{\omega}, y_{\omega})) d\mu(\omega).$$

Let  $\Sigma_o$  be the space of directions at our base point  $o \in \mathbb{H}$ . The tangent cone  $T_o$  is simply the tangent space at  $o$  and thus isometric to  $\mathbb{R}^2$ .

**Proposition 4.7.** (see also [Mon06, Remark 48]) Let  $x$  be a point in  $L^2([0, 1], \mathbb{H})$ . The space of directions at  $x$  is isometric to  $\int_{[0,1]}^* \Sigma_o$ . The tangent cone at  $x$  is isometric to the Pythagorean integral  $L^2([0, 1], T_o)$  which is a Hilbert space.

*Proof.* Let  $g, g'$  be two geodesics rays of  $L^2([0, 1], \mathbb{H})$  starting at  $x$ . Thanks to the description of geodesics, there exist  $\{g_t\}, \{g'_t\}$ , families of geodesic rays starting at  $o$  in  $\mathbb{H}$  and  $v, v'$  measurable maps  $[0, 1] \rightarrow \mathbb{R}^+$  with  $L^2$ -norm equal to 1. Therefore,

$$\begin{aligned}
\cos(\angle_x(g, g)) &= \lim_{s \rightarrow 0} \cos(\overline{\angle}_x(g(s), g(s))) \\
&= \lim_{s \rightarrow 0} \frac{2s^2 - d(g(s), g'(s))^2}{2s^2} \\
&= 1 - 1/2 \lim_{s \rightarrow 0} \frac{d(g(s), g'(s))^2}{s^2} \\
&= 1 - 1/2 \lim_{s \rightarrow 0} \frac{1}{s^2} \int_t (v_t^2 + v'_t{}^2)s^2 - 2v_tv'_t \cos(\overline{\angle}_o(g_t(v_ts), g'_t(v'_ts))) dt \\
&= \int_t v_tv'_t \cos(\angle_o(g_t, g'_t)) dt.
\end{aligned}$$

This equality shows that  $\Sigma_x$  embeds isometrically in  $\int_{[0, 1]}^* \Sigma_o$ . Conversely, if  $((g_t), (v_t))$  is an element in  $\int_{[0, 1]}^* \Sigma_o$ , one can construct the geodesic  $s \mapsto g(s)$  where  $(g(s))_t = g_t(v_ts)$  for almost every  $t$ .

Now, we define a map  $\Phi: T_x \rightarrow L^2([0, 1], T_o)$  through the formula

$$(\lambda, (g_t, v_t)) \mapsto (\lambda v_t, g_t).$$

We compute

$$d((\lambda, (g, v)), (\lambda', (g', v')))^2 = \lambda^2 + \lambda'^2 - 2\lambda\lambda' \int_{[0, 1]} v_tv'_t \cos(\angle_o(g_t, g'_t)) dt$$

and

$$\begin{aligned}
d((\lambda v_t, g_t), (\lambda' v'_t, g'_t))^2 &= \int_{[0, 1]} (\lambda v_t)^2 + (\lambda' v'_t)^2 - 2\lambda v_t \cdot \lambda' v'_t \cos(\angle_o(g_t, g'_t)) dt \\
&= \lambda^2 + \lambda'^2 - 2\lambda\lambda' \int_{[0, 1]} v_tv'_t \cos(\angle_o(g_t, g'_t)) dt.
\end{aligned}$$

This shows that  $\Phi$  is an isometry and its inverse is given by

$$(\lambda_t, g_t) \mapsto (\lambda, (g_t, \lambda_t/\lambda))$$

where  $\lambda = \sqrt{\int \lambda_t^2 dt}$ . □

A notion of *bounded curvature* for geodesic metric spaces has been introduced in [She95]. We give a slightly different definition but equivalent in the case of CAT(0) symmetric spaces. If  $x, y, z$  are distinct points in a CAT(0) space, we denote the area of the comparison Euclidean triangle by  $S_{x,y,z}$ .

**Definition 4.8.** A CAT(0) space  $X$  has *bounded curvature* if for any  $p \in X$ , there exist  $\rho_p, \mu_p > 0$  such that for  $x, y, z \in B(p, \rho_p)$ ,  $y' \in ]x, y]$  and  $z' \in ]x, z]$  we have

$$|\overline{Z}_x(y, z) - \overline{Z}_x(y', z')| \leq \mu_p S_{x, y, z}.$$

In the case where  $d(x, y) = d(x, z) = r$  then  $S_{x, y, z} = \overline{Z}_x(y, z) \frac{r^2}{2}$  and the condition of bounded curvature is

$$\left| 1 - \frac{\angle_x(y, z)}{\overline{Z}_x(y, z)} \right| \leq \frac{\mu_x r^2}{2}.$$

Since we restrict our definition of bounded curvature to CAT(0) spaces, it is actually a lower bound condition on the curvature. This condition is a local condition. If  $M$  is a Riemannian manifold with nonpositive sectional curvature and with locally a uniform lower bound on the sectional curvature then  $M$  has bounded curvature. This is a consequence of Rauch comparison theorem [Lan99, Theorem XI.5.1]. In particular, any Riemannian symmetric space of nonpositive sectional curvature has bounded curvature. Since these spaces are homogeneous, the lower bound of the sectional curvature at any point is actually a global lower bound. Observe that a tree with a vertex of valency larger than 2 does not have bounded curvature.

Now we show that  $X = L^2([0, 1], \mathbb{H})$  does not have bounded curvature. We fix  $r > 0$ ,  $\alpha \in (0, \pi)$  and two geodesic rays starting at  $o$  with an angle equal to  $\alpha$  at  $o$ . For  $0 < \lambda < 1$ , we set  $x_1^\lambda = \rho_1(r/\lambda)$  and  $x_2^\lambda = \rho_2(r/\lambda)$ . We construct points  $x, y^\lambda, z^\lambda \in X$  defined by

$$\begin{aligned} x_t &= o \text{ for } t \in [0, 1], \\ y_t^\lambda &= o \text{ for } t \in (\lambda, 1], \\ z_t^\lambda &= o \text{ for } t \in (\lambda, 1], \\ y_t^\lambda &= x_1^\lambda \text{ for } t \in [0, \lambda], \\ z_t^\lambda &= x_2^\lambda \text{ for } t \in [0, \lambda]. \end{aligned}$$

We have  $d(x, y^\lambda) = d(x, z^\lambda) = r$  and  $\overline{Z}_x(y^\lambda, z^\lambda) = \overline{Z}_o(x_1^\lambda, x_2^\lambda)$  which tends to  $\pi$  as  $\lambda \rightarrow 0$ . Since  $\angle_x(y^\lambda, z^\lambda) = \angle_o(x_1^\lambda, x_2^\lambda) = \alpha$ ; choosing  $\alpha$  small enough, the bounded curvature condition is not satisfied.

If two geodesic rays starting at a point  $x \in X = L^2([0, 1])$  have vanishing Alexandrov angle then they are actually contained one in another. This allows us to define an exponential map  $\exp_x: T_x \rightarrow X$ . If  $v \in T_x$  then  $\exp_x(v)$  is defined to be the point at distance  $\|v\|$  from  $x$  in the direction corresponding to  $v$ . This map is a bijection and its inverse is continuous but the same example as above shows that  $\exp_x$  is not continuous.

*Remark 4.9.* This space has long been known and one can find a similar space denoted  $\mathcal{H}^0(M, M')$  on p.134 of [ES64]. The authors claimed that this space is not a manifold.

*Remark 4.10.* It has been proved in [Duc11, Proposition 3.9] that a CAT(0) symmetric space with bounded curvature and no branching geodesics is homeomorphic to a Hilbert space. More precisely, an exponential map is defined from the tangent cone to the space and this exponential map is a homeomorphism.

## 5 Nonnegative curvature

Orthogonal symmetric Lie algebras of finite dimension play an important role in the theory of finite dimensional Riemannian symmetric spaces. We give the following definition in the context of semisimple  $L^*$ -algebras.

**Definition 5.1.** An *orthogonal symmetric  $L^*$ -algebra* is a pair  $(L, s)$  where

- (i)  $L$  is a real semisimple  $L^*$ -algebra,
- (ii)  $s$  is an involutive isometric automorphism of the  $L^*$ -algebra  $L$ ,
- (iii) For all  $X \in L$  such that  $s(X) = X$ ,  $X^* = -X$ .

In finite dimension, there is a duality between orthogonal symmetric Lie algebras of compact type and orthogonal symmetric Lie algebras of noncompact type (see, e.g., [Hel01, Section V.2]). This duality extends to the context of  $L^*$ -algebras.

Let  $(L_0, s)$  be a symmetric orthogonal  $L^*$ -algebra and let  $\tilde{L}$  be its complexification as  $L^*$ -algebra [Uns71, §1.1]. The automorphism  $s$  extends linearly to a  $L^*$ -automorphism of  $\tilde{L}$ . Let  $L = \mathfrak{k} \oplus \mathfrak{p}$  be the decomposition of  $L$  into  $+1$  and  $-1$  eigenspaces of  $s$ . We set  $L$  to be  $\mathfrak{k} \oplus i\mathfrak{p} \subset \tilde{L}$  and we endow  $L$  with the restriction of  $s$ . This is an orthogonal symmetric  $L^*$ -algebra which we call the *dual* of  $(L, s)$ . Observe that taking the dual is an involutive operation.

A symmetric orthogonal  $L^*$ -algebra  $(L, s)$ , is called *irreducible* if it has no  $s$ -invariant ideal and it has *compact type* (resp. *noncompact type*) if the underlying  $L^*$ -algebra has compact type (resp. noncompact type). The dual of a symmetric orthogonal  $L^*$ -algebra of compact type (resp. of noncompact type) is of noncompact (resp. of compact) type.

**Lemma 5.2.** Let  $(L, s)$  be a symmetric orthogonal  $L^*$ -algebra of noncompact type. For any  $X \in L$ ,  $s(X) = -X^*$ .

*Proof.* Let  $L = \mathfrak{k} \oplus \mathfrak{p}$  be the decomposition of  $L$  into  $-1$  and  $+1$  eigenspaces of  $s$  and let  $L = \mathfrak{k}' \oplus \mathfrak{p}'$  into  $+1$  and  $-1$  eigenspaces of  $s$ . By assumption, we know that  $\mathfrak{k}' \subseteq \mathfrak{k}$  and thus  $\mathfrak{p} \subseteq \mathfrak{p}'$ . Observe that  $[\mathfrak{p}', \mathfrak{p}'] \subseteq \mathfrak{k}'$  and that  $\overline{[\mathfrak{p}, \mathfrak{p}]} = \mathfrak{k}$ . Thus,  $\mathfrak{k} = \overline{[\mathfrak{p}, \mathfrak{p}]} \subseteq \overline{[\mathfrak{p}', \mathfrak{p}']} \subseteq \mathfrak{k}'$ .  $\square$

This previous lemma shows that the notion of orthogonal symmetric algebra is not interesting for  $L^*$ -algebras of noncompact type.

**Proposition 5.3.** *Let  $(L, s)$  be a separable orthogonal symmetric  $L^*$ -algebra of compact type. If  $L = \sum L_i$  is the decomposition of  $L$  into simple ideals then  $s$  permutes the  $L_i$ 's. The algebra  $L$  is the Hilbertian sum of irreducible orthogonal symmetric  $L^*$ -algebras  $I_k$ . Each  $I_k$  is equal to some  $s$ -invariant simple ideal or  $I_k = L_i \oplus L_j$  with  $s(L_i) = L_j$  for some  $L_i$  and  $L_j$ .*

*If  $I_k = L_i \oplus L_j$  with  $s(L_i) = L_j$  then  $L_i$  is isomorphic to  $L_j$  which is isomorphic to  $\mathfrak{o}^2(\infty), \mathfrak{u}^2(\infty)$  or  $\mathfrak{sp}^2(\infty)$ . The decomposition  $I_k = \mathfrak{k} \oplus \mathfrak{p}$  into  $+1$  and  $-1$  eigenspaces of  $s$  is given by  $\mathfrak{k} = \{(X, s(X)); X \in L_i\}$  and  $\mathfrak{p} = \{(X, -s(X)); X \in L_i\}$ .*

*Assume  $L_i$  is  $s$ -invariant. If we decompose  $L_i = \mathfrak{k} \oplus \mathfrak{p}$  into  $+1$  and  $-1$  eigenspaces of  $s$  then  $L_i$  is isomorphic to one orthogonal symmetric  $L^*$ -algebra of the following list :*

Type	$L^*$ -algebra	$\mathfrak{k}$
AI	$\mathfrak{u}^2(\infty)$	$\mathfrak{o}^2(\infty)$
AII	$\mathfrak{u}^2(\infty)$	$\mathfrak{sp}^2(\infty)$
AIII	$\mathfrak{u}^2(p + \infty)$	$\mathfrak{u}^2(p) \times \mathfrak{u}^2(\infty), p \in \mathbb{N}^* \cup \{\infty\}$
BDI	$\mathfrak{o}^2(p + \infty)$	$\mathfrak{o}^2(p) \times \mathfrak{o}^2(\infty), p \in \mathbb{N}^* \cup \{\infty\}$
BDIII	$\mathfrak{o}^2(\infty)$	$\mathfrak{u}^2(\infty)$
CI	$\mathfrak{sp}^2(\infty)$	$\mathfrak{u}^2(\infty)$
CII	$\mathfrak{sp}^2(p + \infty)$	$\mathfrak{sp}^2(p) \times \mathfrak{sp}^2(\infty), p \in \mathbb{N}^* \cup \{\infty\}$

*Remark 5.4.* The above description of simple orthogonal symmetric  $L^*$ -algebras of compact type has the advantage to be brief but it is not explicit. The subalgebra  $\mathfrak{k}$  is given up to isomorphism but the embedding in  $L_i$  and the involution are not given. An explicit description can be obtained in the proof of Proposition 5.3, that is obtained as the dual of some simple  $L^*$ -algebra of noncompact type.

*Proof of Proposition 5.3.* Since  $s$  is  $L^*$ -automorphism, the image of a simple ideal is also a simple ideal. The decomposition  $L = \sum L_i$  is unique up to permutation. Therefore, for any  $i$  there is  $j$  such that  $s(L_i) = L_j$ .

Now it suffices to understand involutive  $L^*$ -automorphisms of simple  $L^*$ -algebras of compact type. Let  $L_0$  be a simple  $L^*$ -algebra of compact type with an involutive  $L^*$ -automorphism  $s$ . We decompose  $L_0 = \mathfrak{k} \oplus \mathfrak{p}$  into  $\pm 1$  eigenspaces of  $s$ . Let  $\tilde{L}$  be the complexification of  $L_0$ . Since  $L_0$  is of compact type,  $L_0$  has no complex structure and thus ([Uns71, Theorem 1.3.1])  $\tilde{L}$  is simple. Let  $L$  be the real form of  $\tilde{L}$  associated to  $s$  (extended to  $\tilde{L}$ ) (see *loc. cit.*). Since  $L_0$  is compact, we know that  $L = \mathfrak{k} \oplus i\mathfrak{p}$ , that is the dual of  $L_0$ . The  $L^*$ -algebra  $L$  is a simple  $L^*$ -algebra of noncompact type and thus is one of those described in section 4.2 or more precisely in [Uns71, Section 5]. Thanks to Lemma 5.2,  $L_0$  is the dual of  $L$  with its unique possible structure of orthogonal symmetric  $L^*$ -algebra.  $\square$

*Proof of Theorem 1.9.* We follow the strategy used in [dR52]. Let  $I$  be an ideal of the  $L^*$ -algebra associated to  $M$  at a point  $p$ . Assume that  $I$  is invariant under  $d_p\sigma_p$ . The

orthogonal of  $I$  is also an ideal of  $L$  invariant under  $d_p\sigma_p$ . We denote it by  $J$ . Thanks to Proposition 4.4, we know that  $E = \exp_p(I \cap \mathfrak{p})$  and  $F = \exp_p(J \cap \mathfrak{p})$  are totally geodesic submanifolds of  $M$ . In particular,  $E$  and  $F$  are symmetric spaces of compact type on their own. The  $L^*$ -algebras associated to them are respectively  $I$  and  $J$ .

Theorem 1.3 yields the existence of some  $r > 0$  and a map  $\phi : B_r(p, E) \times B_r(p, F) \rightarrow M$  which is a diffeomorphism on its image  $U_r$ , which is included in a normal neighborhood of  $p \in M$ . Here  $B_r(p, E)$  and  $B_r(p, F)$  denote the closed balls around  $p$  in  $E$  and  $F$ . We show that the map  $\phi$  can be extended along any geodesic path in  $E \times F$ . This establishes an analog of [dR52, Proposition 1]. Let  $c$  be a geodesic path in  $E \times F$ . There exist two geodesic paths  $c_E$  and  $c_F$  such that  $c = c_E \times c_F$ . Let  $q = (q_E, q_F)$  be the extremity of  $c$ . Let  $x = (x_E, x_F)$  be the point  $c(r)$ . We denote by  $\tau_{p,x}$  (resp.  $\tau_{x_E,p}$ ,  $\tau_{x_F,p}$ ) the (unique) transvection from  $p$  to  $x$  (resp. from  $x_E$  to  $p$  and  $x_F$  to  $p$ ). The map  $\tau_{p,x} \circ \phi \circ (\tau_{x_E,p} \times \tau_{x_F,p})$  is an isometry from  $B_r(x_E, E) \times B_r(x_F, F)$  to  $\tau_{p,x}(U_r)$ . Moreover, this map coincides with  $\phi$  on  $(B_r(x_E, E) \times B_r(x_F, F)) \cap (B_r(p, E) \times B_r(p, F))$ . Repeating this procedure at most  $\lceil l/r \rceil$  times, we obtain an extension of  $\phi$  along  $c$ . We can now conclude as in [dR52, §7] that  $E$  and  $F$  are also simply connected and  $M$  is isometric to  $E \times F$ .

Let  $L = \sum I_i$  be the decomposition of  $L$  into irreducible ideals invariant under  $d_p\sigma_p$ . That is,  $I_i$  is a simple ideal invariant under  $d_p\sigma_p$  or  $I_i = L_1 \oplus L_2$  where  $L_1, L_2$  are simple ideals interchanged by  $d_p\sigma_p$ . For  $n \in \mathbb{N}$ , we set  $I'_n = \sum_{i>n} I_i$ . Let  $N_i = \exp_p(I_i \cap \mathfrak{p})$  for  $i \in \mathbb{N}$  and let  $N'_n = \exp_p(I'_n \cap \mathfrak{p})$ . Those manifolds are simply connected Riemannian spaces of compact type and an induction on  $n$  shows that  $M$  is isomorphic to  $N_1 \times \cdots \times N_n \times N'_n$ . The same density argument as in the proof of Theorem 1.7 shows that  $M$  is isometric to  $\prod_i^2 N_i$ .

Each  $N_i$  is completely determined by the  $L^*$ -algebra associated to it. Actually, if  $M_i$  is a simply connected Riemannian space with same associated  $L^*$ -algebra, then the above extension argument shows that the local isomorphism obtained in Theorem 1.3 between two open subsets of  $N_i$  and  $M_i$  can be extended to a global isomorphism.  $\square$

## References

- [Bal72] V. K. Balachandran. Real  $L^*$ -algebras. *Indian J. Pure Appl. Math.*, 3(6):1224–1246, 1972.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [Bor01] Armand Borel. *Essays in the history of Lie groups and algebraic groups*, volume 21 of *History of Mathematics*. American Mathematical Society, Providence, RI, 2001.

- [Bou87] N. Bourbaki. *Topological vector spaces. Chapters 1–5*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1987. Translated from the French by H. G. Eggleston and S. Madan.
- [CL10] Pierre-Emmanuel Caprace and Alexander Lytchak. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.*, 346(1):1–21, 2010.
- [CM09] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *J. Topol.*, 2(4):661–700, 2009.
- [dlH71] Pierre de la Harpe. Classification des  $L^*$ -algèbres semi-simples réelles séparables. *C. R. Acad. Sci. Paris Sér. A-B*, 272:A1559–A1561, 1971.
- [dlH72] Pierre de la Harpe. *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space*. Lecture Notes in Mathematics, Vol. 285. Springer-Verlag, Berlin, 1972.
- [dR52] Georges de Rham. Sur la reductibilité d’un espace de Riemann. *Comment. Math. Helv.*, 26:328–344, 1952.
- [Duc11] Bruno Duchesne. *Des espaces de Hadamard symétriques de dimension infinie et de rang fini*. PhD thesis, Université de Genève, Juillet 2011.
- [Duc12] Bruno Duchesne. Infinite dimensional non-positively curved symmetric spaces of finite rank. *International Mathematical Research Notices*, 2012, doi: 10.1093/imrn/rns093.
- [ES64] James Eells, Jr. and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.
- [GM75] S. Gallot and D. Meyer. Opérateur de courbure et laplacien des formes différentielles d’une variété riemannienne. *J. Math. Pures Appl. (9)*, 54(3):259–284, 1975.
- [Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.
- [Kau81] Wilhelm Kaup. Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. I. *Math. Ann.*, 257(4):463–486, 1981.
- [Kau83] Wilhelm Kaup. Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. II. *Math. Ann.*, 262(1):57–75, 1983.
- [KL97] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.



- [Kli95] Wilhelm P. A. Klingenberg. *Riemannian geometry*, volume 1 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 1995.
- [Lan99] Serge Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
- [Lar07] Gabriel Larotonda. Nonpositive curvature: a geometrical approach to Hilbert-Schmidt operators. *Differential Geom. Appl.*, 25(6):679–700, 2007.
- [Mca65] John Harris Mcalpin. *INFINITE DIMENSIONAL MANIFOLDS AND MORSE THEORY*. ProQuest LLC, Ann Arbor, MI, 1965. Thesis (Ph.D.)—Columbia University.
- [Mon06] Nicolas Monod. Superrigidity for irreducible lattices and geometric splitting. *J. Amer. Math. Soc.*, 19(4):781–814, 2006.
- [Pet06] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.
- [Sch60] John R. Schue. Hilbert space methods in the theory of Lie algebras. *Trans. Amer. Math. Soc.*, 95:69–80, 1960.
- [Sch61] John R. Schue. Cartan decompositions for  $L^*$  algebras. *Trans. Amer. Math. Soc.*, 98:334–349, 1961.
- [She95] B. U. Shergoziev. Infinite-dimensional spaces with bounded curvature. *Sibirsk. Mat. Zh.*, 36(5):1167–1178, iv, 1995.
- [Sim62] James Simons. On the transitivity of holonomy systems. *Ann. of Math. (2)*, 76:213–234, 1962.
- [Tum09] Alice Barbara Tumpach. On the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits. *Forum Math.*, 21(3):375–393, 2009.
- [Uns71] Ignacio Unsain. Classification of the simple separable real  $L^*$ -algebras. *Bull. Amer. Math. Soc.*, 77:462–466, 1971.